Colorings of b-simple hypergraphs

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This work has an aim to estimate the maximum edge degree such that any n-uniform b-simple hypegraph with lower maximum edge degree can be colored properly using r-colours.

Recall some basic definitions. Hypergraph is a pair (V, E) where V is a set, called the *vertex set* of the hypergraph and E is a family of subsets of V, whose elements are called the *edges* of the hypergraph. A hypergraph is *n-uniform* if every of its edges contains exactly n vertices. The *degree of an edge A* in a hypergraph H is the number of other edges of H which have nonempty intersection with A. The maximum edge degree of H is denoted by $\Delta(H)$.

An r-coloring of hypergraph $H = (V, E)$ is a mapping from the vertex set V to the set of r colors, $\{0, \dots, r-1\}$. A coloring of H is called *proper* if it does not create monochromatic edges (i.e. every edge contains at least two vertices which receives different colors). A hypergraph is said to be *r-colorable* if there exists a proper r-coloring of that hypergraph.

Consider the family of b-simple hypergraphs, in which any two edges do not share more than b common vertices. The best known result is due to Kozik [1] who showed that for any b-simple n-uniform hypergraph H, the condition

$$\Delta(H) \leqslant c(b, r)\frac{n}{\ln n} r^n \qquad (0.1)$$

implies the r-colorability of H, where $c(r, b) > 0$ is some positive function of r and b.

The main result of the paper refines the estimate (0.1) as follows.

Theorem 1. *Suppose $b \geqslant 1$, $r \geqslant 2$ and $n > n_0(b)$ is large enough in comparison with b. Then if a b-simple n-uniform hypergraph H satisfies the inequality*

$$\Delta(H) \leqslant c \cdot n\, r^{n-b}, \qquad (0.2)$$

where $c > 0$ is some absolute constant, then H is r-colorable.

In the case of simple hypergraphs, i.e. for $b = 1$, the above result (0.2) is not new. It was obtained previously by Kozik and Shabanov [2]. For fixed r, b, the bound (0.2) is $\Theta_{r,b}(n)$ times smaller than the known upper bound proved by Kostochka and Rödl [3].

References

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[3] A.V. Kostochka, V. Rödl, "Constructions od sparse uniform hypergraphs with high chromatic number", *Random Structures and Algorithms*, **36**:1 (2010), 46–56.

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