# Random permutations with cycle weights

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We deal with random permutations $\sigma$ sampled from the symmetric group $\mathbf{S}_n$ according to the generalized Ewens probability measure defined by

$$\nu_{n,\Theta}(\{\sigma\}) := \frac{1}{n!m(n)} \prod_{j\le n} \theta_j^{k_j(\sigma)}, \qquad m(n) := \sum_{1s_1+\cdots+ns_n=n} \prod_{j=1}^{n} \Big(\frac{\theta_j}{j}\Big)^{s_j} \frac{1}{s_j!},$$

where $k_j(\sigma) \in \mathbf{N}_0$ is the number of cycles of length $j$ in $\sigma$ and $\theta_j \ge 0$ are arbitrary weights. Summation in the last formula runs over $s_j \in \mathbf{N}_0$, $1 \le j \le n$. One purpose of the talk is to discuss the distribution of the ordered statistics $J_1(\sigma) \ge J_2(\sigma) \ge \cdots J_w(\sigma) \ge 1$ of all cycle lengths in $\sigma$ listed according to their multiplicities. Namely, if $Y_n := n^{-1}\big(J_1, J_2, \dots, J_w, 0, \dots\big)$, where $J_i := J_i(\sigma)$, and $P_{n,\Theta} = \nu_{n,\Theta} \cdot Y_n^{-1}$ is the distribution, then we seek conditions under which the latter converges to a limit measure concentrated in the simplex

$$\Big\{(x_1, x_2, \dots) \in [0,1]^{\mathbf{N}} : \ x_1 \ge x_2 \ge \cdots, \ x_1 + x_2 + \cdots = 1\Big\}.$$

as $n \to \infty$. So far, the most general result was obtained if $\theta_j \to \theta$ as $j \to \infty$. Ours sounds as follows.

**Theorem.** *If $\theta_j \le C < \infty$, $j \ge 1$,*

$$\frac{1}{n}\sum_{j\le n} \theta_j \to \theta > 0$$

*and $n \to \infty$, then $P_{n,\Theta}$ weakly converges to the Poisson-Dirichlet distribution $PD(\theta)$.*

The extension already covers the so-called $\mathbf{A}$-permutations, that is, permutations whose cycles lengths belong to a set $\mathbf{A} \subset \mathbf{N}$ having the natural asymptotic density $\theta$. The result is invariant under the shift $\theta_j \mapsto \rho^j\theta_j$ with an arbitrary constant $\rho > 0$. Our proof is based upon the fact that the given conditions imply regularity of the sequence $m(n)$ as $n \to \infty$. The latter is established combining a Tauberian theorem with the comparative analysis of power series coefficients cultivated in recent author's papers. An historical account of the problem can be found in Section 5.7 of the monograph by R. Arratia, A.D. Barbour and S. Tavaré (*Logarithmic Combinatorial Structures: A Probabilistic Approach*, EMS Publishing House, Zürich, 2003).