On limit points of spectra of first order graph properties with small quantifier depth

A.D. Matushkin

The random graph $G(n,p)$ *obeys zero-one law w.r.t. a first order sentence* ϕ, if either a.a.s. $G(n,p) \models \phi$, or a.a.s. $G(n,p) \models \neg(\phi)$. For each first order sentence ϕ consider the set of all $\alpha > 0$ such that $G(n, n^{-\alpha})$ does not obey zero-one law w.r.t. ϕ. This set is called *the spectrum of* ϕ. In 1988 [1], S. Shelah and J. Spencer proved that all points of $S(\phi)$ are rational numbers. In 1990 [2], J. Spencer proved that there exists a first order sentence with an infinite spectrum and the quantifier depth 14. Let $q_{\min}$ be the minimal quantifier depth of a first order sentence with an infinite spectrum. The best known upper bound for $q_{\min}$ is $5 \geq q_{\min}$ (see [3]).

Theorem 1. *There exists a first order sentence* ϕ *with quantifier depth 5 whose spectrum contains all the numbers* $\alpha = \frac{1}{2} + \frac{1}{2(m+1)}$, $m \in \mathbb{N}$.

In 2012 [4], M. Zhukovskii proved that for any first order sentence ϕ with the quantifier depth k the set $S(\phi) \cap (0, 1/(k-2))$ is finite. Later [5], it was proved that the set $S(\phi) \cap (1, \infty)$ is also finite. In particular, for any ϕ with the quantifier depth 3, $S(\phi) \cap (0,1) = \varnothing$ (so $q_{\min} \geq 4$), and for any ϕ with the quantifier depth 4, all limit points of $S(\phi)$ must be in $[1/2, 1)$.

So, the exact value of $q_{\min}$ is unknown, but we know that $q_{\min} \in \{4, 5\}$. Denote by $S(k)$ the union of all $S(\phi)$ for all ϕ with the quantifier depth k. We examined the set $S(4)$ and proved that it has no limit points except possibly the points 1/2 and 3/5. We also proved that the spectrum of first order sentences, whose sequences of nested quantifiers are all of form $\forall\exists\forall\exists$ or $\exists\forall\exists\forall$, is finite.

References

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