On limit points of spectra of first order graph properties with small quantifier depth

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The random graph $G(n,p)$ obeys zero-one law w.r.t. a first order sentence $\phi$, if either a.a.s. $G(n,p) \models \phi$, or a.a.s. $G(n,p) \models \neg(\phi)$. For each first order sentence $\phi$ consider the set of all $\alpha > 0$ such that $G(n,n^{-\alpha})$ does not obey zero-one law w.r.t. $\phi$. This set is called the spectrum of $\phi$. In 1988 [1], S. Shelah and J. Spencer proved that all points of $S(\phi)$ are rational numbers. In 1990 [2], J. Spencer proved that there exists a first order sentence with an infinite spectrum and the quantifier depth 14. Let $q_{\text{min}}$ be the minimal quantifier depth of a first order sentence with an infinite spectrum. The best known upper bound for $q_{\text{min}}$ is $5 \geq q_{\text{min}}$ (see [3]).

Theorem 1. There exists a first order sentence $\phi$ with quantifier depth 5 whose spectrum contains all the numbers $\alpha = \frac{1}{2} + \frac{1}{2(m+1)}$, $m \in \mathbb{N}$.

In 2012 [4], M. Zhukovskii proved that for any first order sentence $\phi$ with the quantifier depth $k$ the set $S(\phi) \cap (0,1/(k-2))$ is finite. Later [5], it was proved that the set $S(\phi) \cap (1,\infty)$ is also finite. In particular, for any $\phi$ with the quantifier depth 3, $S(\phi) \cap (0,1) = \emptyset$ (so $q_{\text{min}} \geq 4$), and for any $\phi$ with the quantifier depth 4, all limit points of $S(\phi)$ must be in $[1/2,1)$.

So, the exact value of $q_{\text{min}}$ is unknown, but we know that $q_{\text{min}} \in \{4,5\}$. Denote by $S(k)$ the union of all $S(\phi)$ for all $\phi$ with the quantifier depth $k$. We examined the set $S(4)$ and proved that it has no limit points except possibly the points 1/2 and 3/5. We also proved that the spectrum of first order sentences, whose sequences of nested quantifiers are all of form $\forall \exists \forall \exists$ or $\exists \forall \exists \forall$, is finite.

References