On the j-chromatic number of random hypergraphs

Alexander Semenov*

Let $H = (V, E)$ be a hypergraph. A hypergraph H is k-uniform if all edges of H are of size k. A random k-uniform hypergraph $H(n, k, p)$ is a k-uniform hypergraph on n labeled vertices $V = \{v_1, \dots, v_n\}$, in which every subset $e \subset V$ of size k is chosen to be an edge of H randomly and independently with probability p. We will study the chromatic number of random hypergraphs. Actually, a family of chromatic numbers can be defined.

Definition 1. *For an integer j, a j-independent set in a hypergraph $H = (V, E)$ is a subset $W \subset V$ such that for every edge $e \in E : |e \cap W| \leqslant j$.*

Definition 2. *A j-proper coloring of $H = (V, E)$ is a partition of the vertex set V of H into disjoint union of j-independent sets, so called colors. The j-chromatic number $\chi_j(H)$ of H is the minimal number of colors needed for a j-proper coloring of H.*

The main interest of this work is the asymptotic behavior of the property of hypergraph $H(n, k, p)$ to have its j-chromatic number equal to 2. By asymptotic properties of $H(n, k, p)$ we consider n as tending to infinity while k and j are kept constant.

It can be showed that the previously mentioned property of random hypergraph has a sharp threshold [1]. The case of $j = k - 1$ was intensively studied and authors of [2] have found the upper and lower bound for that threshold but there was a large gap between those bounds. Later in works [3], [4] and [5] bounds were improved and the gap was reduced to the $O_k(1)$.

Here we consider the generalization to the case when j is less than $k - 1$. Main result is showed in a theorem below

Theorem 1. *Suppose $1 < k - j \leqslant \varphi(k)$, where $\varphi(k) = o(k^{1/2})$. There exists $k_0 \in \mathbb{N}$, such that if $k > k_0$ and*

$$c > \frac{2^{k-1} \ln 2}{\sum\limits_{i=j+1}^{k} \binom{k}{i}} - \frac{\ln 2}{2} + O\left(2^{1-k} k^{k-j-1}\right)$$

*Moscow state university; Moscow institute of physics and technology.

then w.h.p. as n *tends to infinity,* $\chi_j\left(H(n,k,cn/\binom{n}{k})\right) > 2$. *Otherwise, if*

$$c < \frac{2^{k-1}\ln 2}{\sum\limits_{i=j+1}^{k}\binom{k}{i}} - \frac{\ln 2}{2} + O(k^{j+1-k})$$

then w.h.p. as n *tends to infinity,* $\chi_j\left(H(n,k,cn/\binom{n}{k})\right) \leqslant 2$.

As reader can see, in comparison with the case $j = k-1$ the gap between upper bound and lower bound in the theorem tends to zero with growth of k.

References

[1] H. Hatami, M. Molloy, "Sharp thresholds for constraint satisfaction problems and homomorphisms", *Random Structures and Algorithms, 33:3 (2008)*, 310-332.

[2] N. Alon, J. Spencer, "A note on coloring random k-sets", Unpublished manuscript.

[3] D. Achlioptas, J.H. Kim, M. Krivelevich, P. Tetali, "Two-colorings random hypergraphs", *Random Structures and Algorithms, 20:2 (2002)*, 249-259.

[4] D. Achlioptas, C. Moore, "Random k-SAT: two moments suffice to cross a sharp threshold", *SIAM Journal on Computing, 36:3 (2005)*, 740-762.

[5] A. Coja-Oghlan, L. Zdeborova, "The condensation transition in random hypergraph 2-coloring", *Proc. 23rd Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, 2012*, 241-250.