On the $j$-chromatic number of random hypergraphs

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Let $H = (V, E)$ be a hypergraph. A hypergraph $H$ is $k$-uniform if all edges of $H$ are of size $k$. A random $k$-uniform hypergraph $H(n, k, p)$ is a $k$-uniform hypergraph on $n$ labeled vertices $V = \{v_1, \ldots, v_n\}$, in which every subset $e \subset V$ of size $k$ is chosen to be an edge of $H$ randomly and independently with probability $p$. We will study the chromatic number of random hypergraphs. Actually, a family of chromatic numbers can be defined.

**Definition 1.** For an integer $j$, a $j$-independent set in a hypergraph $H = (V, E)$ is a subset $W \subset V$ such that for every edge $e \in E$:
\[ |e \cap W| \leq j. \]

**Definition 2.** A $j$-proper coloring of $H = (V, E)$ is a partition of the vertex set $V$ of $H$ into disjoint union of $j$-independent sets, so called colors. The $j$-chromatic number $\chi_j(H)$ of $H$ is the minimal number of colors needed for a $j$-proper coloring of $H$.

The main interest of this work is the asymptotic behavior of the property of hypergraph $H(n, k, p)$ to have its $j$-chromatic number equal to 2. By asymptotic properties of $H(n, k, p)$ we consider $n$ as tending to infinity while $k$ and $j$ are kept constant.

It can be showed that the previously mentioned property of random hypergraph has a sharp threshold [1]. The case of $j = k - 1$ was intensively studied and authors of [2] have found the upper and lower bound for that threshold but there was a large gap between those bounds. Later in works [3], [4] and [5] bounds were improved and the gap was reduced to the $O_k(1)$.

Here we consider the generalization to the case when $j$ is less than $k - 1$. Main result is showed in a theorem below

**Theorem 1.** Suppose $1 < k - j \leq \varphi(k)$, where $\varphi(k) = o(k^{1/2})$. There exists $k_0 \in \mathbb{N}$, such that if $k > k_0$ and
\[
 c > \frac{2^{k-1} \ln 2}{k} \sum_{i=j+1}^{k} \binom{k}{i} - \frac{\ln 2}{2} + O \left( 2^{1-k} k^{k-j-1} \right)
\]

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then w.h.p. as $n$ tends to infinity, $\chi_j(H(n,k, cn/(n^k))) > 2$. Otherwise, if

$$c < \frac{2^{k-1} \ln 2}{\sum_{i=j+1}^{k}\binom{k}{i}} - \frac{\ln 2}{2} + O(k^{j+1-k})$$

then w.h.p. as $n$ tends to infinity, $\chi_j(H(n,k, cn/(n^k))) \leq 2$.

As reader can see, in comparison with the case $j = k - 1$ the gap between upper bound and lower bound in the theorem tends to zero with growth of $k$.

References


