Uniqueness of vertices with minimal r-neighbourhoods in a random graph

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In the random graph $G(n, p)$ the size of the first neighbourhood (degree) of a vertex is binomially distributed. Likewise, the size of the r-neighbourhood of vertex is also distributed binomially. However, the two parameters of this binomial distribution depend, in a non trivial way, on size of all the other k-neighbourhoods for $k = 1, \ldots, r-1$. This is a barrier to writing down a simple expression for the probability that the r-neighbourhood of a vertex has size k. We derive a simple asymptotic formula for the probability of this event by the Laplace method and then use this expression to prove the following theorem: Let $0 < p(n) < 1$ and $r(n) \geq 2$ be such that $(np)^{2r+1} = o(n)$. Then $G(n, p)$ has a unique vertex which attains an r-neighbourhood of minimum size with high probability if and only if $np - \log(n) \to \infty$.